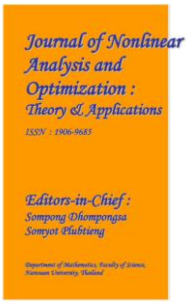


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# FORMULATION OF FINITE ELEMENT METHODS AND TRANSFORMATION FOR HIGHER ORDER TRIANGULAR ELEMENTS

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**Abstract**
The Finite Element Method (FEM) is a very effective technique that is widely used for solving problems in structural mechanics, fluid dynamics, heat transfer and other areas of engineering and science by solving Partial Differential Equations. This paper describes the formulation of the Finite Element Method along with the detailed steps of formulation. We have also shown the Step-by-Step Derivation of Iso-parametric Coordinate Transformation for Higher- order Triangular Elements, discussing their advantages and applications. A method is also shown to determine the points along the curved boundary and the interior of a triangle.

**Keywords**
- Finite Element Method
- Structural Mechanics
- Partial Differential Equations
- Iso-Parametric Coordinate Transformation

## Introduction
Finite Element Formulation refers to the mathematical and computational approach used to solve partial differential equations (PDEs) by dividing a complex domain into smaller, simpler pieces, called finite elements. The finite element method (FEM) is a powerful numerical technique widely used for solving problems in structural mechanics, fluid dynamics, heat transfer, and other areas of engineering and science.

**Steps in Finite Element Formulation**

1. Discretization of the Domain (Meshing): The first step is to discretize the entire problem domain (geometry) into a collection of sub-domains known as "elements." These elements can take various shapes, such as triangles (2D), tetrahedrons (3D) or quadrilaterals (2D).The points at the corners or within the elements are called "nodes."
2. Selection of Element Type and Shape Functions: The next step is to select appropriate interpolation functions (also called shape functions or basis functions) to approximate the solution over each element. These shape functions are typically polynomials (e.g., linear, quadratic) defined in terms of the nodal values.
3. Derivation of Element Equations: Governing equations (typically PDEs like the heat equation, wave equation, etc.) are rewritten in a form suitable for FEM, often using a weak formulation such as the Galerkin Method or Variational Methods.The PDEs are then transformed into a system of algebraic equations for each element. These equations relate the nodal values of the unknown solution to the shape functions.
4. Assembly of Global System of Equations: The local element equations are combined to form a global system of equations that describes the entire problem domain. This involves assembling a stiffness matrix (or other relevant system matrices), load vectors, and boundary conditions.
5. Application of Boundary Conditions: Boundary conditions (such as fixed supports, prescribed displacements, or forces) are applied to the global system of equations.
6. Solving the System of Equations: The assembled system of linear or nonlinear equations is

solved using numerical methods, such as Gaussian Elimination, Iterative Solvers (e.g., Conjugate Gradient), or Direct Solvers (e.g., LU Decomposition).

7. Post-processing: The solution at each node is obtained and interpolated over the elements using the shape functions. Results like stresses, strains, velocities, or heat fluxes can be derived from the nodal solutions for further analysis and visualization.

**Advantages of FEM**

- Flexibility: It can handle complex geometries, boundary conditions, and material properties.
- General applicability: FEM can be used to solve a wide range of problems, from structural analysis to fluid flow.
- Accuracy: By refining the mesh (increasing the number of elements), FEM can achieve high accuracy in solutions.

**Applications**

- Structural Analysis: Used to determine the deformation and stresses in materials and structures.
- Fluid Dynamics (CFD): Used to simulate the flow of fluids and solve problems involving fluid-structure interaction.
- Heat Transfer: Used for simulating heat conduction, convection, and radiation in various media.

To find the equations for point transformation of higher order triangular elements using iso-parametric coordinate transformation. The iso-parametric coordinate transformation is commonly used in Finite Element Methods (FEM) to map elements from their natural (or reference) coordinate system to the global (or real-world) coordinate system. For higher-order triangular elements, the shape functions are more complex than for linear elements, but the principle remains the same.

In this case, we will discuss the transformation of higher-order triangular elements (e.g., quadratic or cubic elements) using iso-parametric transformation. The natural coordinate system for triangular elements uses area coordinates (barycentric coordinates), typically denoted as $(\xi,\eta)$.

**Step-by-Step Derivation of Iso-parametric Coordinate Transformation for Higher-order Triangular Elements**

1. Natural Coordinates for a Triangular Element: A triangular element in its natural coordinate system is usually defined in terms of barycentric (area) coordinates, denoted as $(\xi,\eta)$. These coordinates are constrained as:

$0\leq\xi\leq1,\ 0\leq\eta\leq1,\ \xi+\eta\leq1$

For higher-order elements, we need additional internal nodes.

2. Global Coordinates of the Triangular Element: The physical (global) coordinates of the triangular element are denoted as $(x,y)$. The goal is to establish a transformation between the natural coordinates $(\xi,\eta)$ and the global coordinates $(x,y)$.

3. Quadratic Shape Functions (for 6-node Triangular Element): For a higher-order (quadratic) triangular element with 6 nodes, the global coordinates $(x,y)$ can be expressed as a function of the natural coordinates $(\xi,\eta)$ using iso-parametric shape functions.The shape functions for a 6-node triangular element in natural coordinates $(\xi,\eta)$ are

$N_1=1-3\xi-3\eta+2\xi^2+4\xi+2\eta^2 N_2=\xi(2\xi-1)$

$N_3=\eta(2\eta-1)NN_4=4\xi(1-\xi-\eta)$

$N_5=4\xi\eta N_6=4\eta(1-\eta)$

These shape functions are quadratic polynomials in $(\xi,\eta)$ and are designed to interpolate values at the 6 nodes of the quadratic triangular element.

4. Iso-parametric Coordinate Transformation: The coordinates $(x,y)$in the global coordinate system are related to the natural coordinates $(\xi,\eta)$ using the shape functions Ni and the nodal coordinates $(x_i,y_i)$ as:

$x(\xi,\eta)=\sum i=1\sum 6N_i(\xi,\eta)y_i$

where $x_i$ and $y_i$ are the global coordinates of the 6 nodes of the triangular element, and $N_i(\xi,\eta)$ are the shape functions corresponding to each node.

5. Jacobian Matrix for Coordinate Transformation: To transform between natural and global coordinates, we need to compute the Jacobian matrix, which relates the differential elements in natural and global coordinates. The Jacobian matrix J is defined as

$$J = \begin{matrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{matrix}$$

The partial derivatives of x and y with respect to ξ and η can be computed as

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{6} \frac{\partial N_i}{\partial \xi} x_i \ , \quad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{6} \frac{\partial N_i}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{6} \frac{\partial N_i}{\partial \xi} y_i \ , \quad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{6} \frac{\partial N_i}{\partial \eta} y_i$$

These derivatives give the entries of the Jacobian matrix.

**Element Stiffness Matrix:** The Jacobian is used to transform the integration over the natural coordinates to the global coordinates. For higher-order triangular elements, the element stiffness matrix is computed using Gaussian quadrature over the natural coordinates. The transformed stiffness matrix involves integrating the strain-displacement matrix B and material property matrix D over the element.

The element stiffness matrix Ke is given by

$$K_e = \int_{\Omega} B^T D B \, d\Omega$$

where B is the strain-displacement matrix (transformed using the Jacobian), D is the material property matrix, and Ω is the area of the triangular element in global coordinates.

The iso-parametric coordinate transformation is commonly used in Finite Element Methods (FEM) to map elements from their natural (or reference) coordinate system to the global (or real- world) coordinate system. For higher-order triangular elements, the shape functions are more complex than for linear elements, but the principle remains the same.In this case, we will discuss the transformation of higher-order triangular elements (e.g., quadratic or cubic elements) using iso-parametric transformation. The natural coordinate system for triangular elements uses area coordinates (barycentric coordinates), typically denoted as (ξ,η).

**To determine the points along the Curved Boundary and the Interior of a Triangle**

To determine points along the curved boundary and the interior of a triangular element in the context of finite element analysis (FEA), particularly when dealing with higher-order triangular elements, the following process can be used. The process assumes we have a curved boundary formed using a higher-order representation, such as with quadratic or cubic shape functions.

For this, we will

- Use Iso-parametric Coordinates for Point Representation.
- Define Shape Functions for Higher-order Triangular Elements.
- Map Points from the Natural Coordinate System to the Global Coordinate System.
- Evaluate Points Along the Curved Boundary and Inside the Element.

**Steps to determine points along the Curved Boundary and Interior**

1. **Natural Coordinates (Barycentric Coordinates):** The triangular element can be described in a local (natural) coordinate system, using barycentric coordinates (ξ,η) for parametric representation. These coordinates range from 0 to 1 and are constrained by ξ+η≤1, meaning the element occupies a triangular region in (ξ,η)space.

For example:

Vertices of the triangle are located at:

Vertex 1 at (ξ,η)=(0,0)

Vertex 2 at (ξ,η)=(1,0)

Vertex 3 at (ξ,η)=(0,1)

2. **Higher-Order Shape Functions:** For a quadratic triangular element with 6 nodes (3 corner nodes and 3 mid-side nodes), the global coordinates (x,y) of any point within the element can be expressed using shape functions $N_i(\xi,\eta)$ and the nodal coordinates $(x_i,y_i)$. The shape functions for a quadratic triangular element are

$$N_1 = 1 - 3\xi - 3\eta + 2\xi^2 + 4\xi\eta + 2\eta^2$$

$N_2=\xi(2\xi-1)$

$N_3=\eta(2\eta-1)$

$N_4=4\xi(1-\xi-\eta)$

$N_5=4\xi\eta$

$N_6=4\eta(1-\xi-\eta)$

For a cubic triangular element (with more internal nodes), the shape functions would be of higher degree, but the process is similar.

3. **Global Coordinate Mapping:** The global coordinates (x,y) of a point inside the triangle (or on the boundary) are obtained using the following transformation

$x(\xi, \eta) = nN_i(\xi, \eta)x_i$

$y(\xi, \eta) = nN_i(\xi, \eta)y_i$

Here, n is the number of nodes for the element (e.g., n=6n =6 for a quadratic triangular element), and $(x_i,y_i)$ are the global coordinates of the corresponding nodes.

- Interior points can be calculated by selecting values of (ξ,η) within the triangular domain.
- Boundary points are determined by selecting specific values of ξ and η corresponding to the edges (where one of the coordinates is zero or one).

For example:

- The edge between Node 1 and Node 2 corresponds to η=0and varying ξ between 0 and 1.
- The edge between Node 1 and Node 3 corresponds to ξ=0 and varying η between 0 and 1.
- The edge between Node 2 and Node 3 corresponds to ξ+η=1, with ξ,η≥0.

4. **Points along the Curved Boundary:** For curved boundaries, such as those represented by quadratic shape functions along an edge, the mid-side nodes introduce curvature. To find points along this curve. Parameterize the boundary edge using the natural coordinates. For example, on the edge between Node 1 and Node 2

$\eta=0,\xi\in[0,1]$

Use the shape functions $N_i(\xi,\eta)$ for the corresponding edge nodes (including the mid-side node) to determine the global coordinates (x,y) along the curve

$x(\xi)=N_1(\xi,0)x_1+N_2(\xi,0)x_2+N_4(\xi,0)x_4$

$y(\xi)=N_1(\xi,0)y_1+N_2(\xi,0)y_2+N_4(\xi,0)y_4$

Similarly, we can evaluate the curved boundary for other edges.

5. **Points inside the Triangular Element:** To compute points inside the element, select various pairs of (ξ,η) values such that ξ+η≤1

For each selected (ξ,η) pair, use the global mapping equations.

$x(\xi,\eta)=\sum i=\sum nN_i(\xi,\eta)x_i y(\xi,\eta)=\sum i=N_i(\xi,\eta)y_i$

This will give us the coordinates of points within the triangle. To generate a grid of points inside the element, we can systematically vary ξ and η values, ensuring that ξ+η≤1.

Example: Quadratic Triangle with Curved Edges

Consider a quadratic triangle positioned as follows:

- Node1: $(x_1, y_1)$
- Node2: $(x_2, y_2)$
- Node3: $(x_3, y_3)$
    - Mid-side Node 4(on edge between Node1 and 2): $(x_4, y_4)$
- Mid-side Node 5(on edge between Node2 and 3): $(x_5, y_5)$
- Mid-side Node 6(on edge between Node1 and 3): $(x_6, y_6)$

1. Boundary Points: Along the boundary between Node1and Node2

$x(\xi)=N_1(\xi,0)x_1+N_2(\xi,0)x_2+N_4(\xi,0)x_4$

$y(\xi)=N_1(\xi,0)y_1+N_2(\xi,0)y_2+N_4(\xi,0)y_4$

Similar equations apply for the other edges.

2. Interior Points: Select values of ξ and η such that ξ+η≤1, and apply the shape functions to compute the interior points.

**Finding a Point along a Curved Body:** In FEM, curves are often described parametrically or using finite elements to approximate the curve. To find a point on a curved body, we can use either parametric equations or the FEM approach for curves.

**Parametric Curves:** A curve in 2D or 3D can be described by parametric equations

*r(t)=⟨x'(t), y'(t), z'(t)⟩*
where *t* is the parameter that varies along the curve.
Finding a Point for a given Arc Length
To find a point at specific arc lengths along the curve:
1. Find the Differential of the Curve: Calculate the derivative of the curve with respect to the parameter *t*:
*r′(t)=⟨x′(t),y′(t),z′(t)⟩*
2. Calculate the Arc Length: The arc length from $t_o$ to *t* is given by:
*s=∫tr′(t)dt=∫t∫t0t(x′(t))*
3. Solve for the Parameter *t*: For a given arc length, solve the arc length equation to find the parameter *t*.
4. Find the Point: Substitute the value of *t* into the parametric equations to find the point along the curve.
Example: Arc Length along a 2D Curve
Consider the curve $r(t)=\langle t,t_2\rangle$, a parabola.
1. Derivative of the Curve: *r′(t)=⟨1,2t⟩*
2. Arc-Length: The arc length from $t_0$=0 to *t* is:
$s=\int t_1+(2t)2dt$
This is a standard integral that can be solved to give the arc length.
3. Solve for *t*: For a given arc length, solve this equation for *t*.
4. Find the Point: Once *t* is determined, the corresponding point on the curve is r(t)=⟨t,t2⟩.

**Conclusion**
These methods describe how to solve higher-order differential equations using FEM and how to find a point along a curved body using parametric descriptions and arc length. The paper also shows the method to determine the points along the Curved Boundary and the Interior of a Triangle by taking various examples. Additionally, we also find a step-by-Step Derivation of Iso-parametric Coordinates.

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